

STRONG CONNECTIVITY AND ITS APPLICATIONS

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Abstract. Directed graphs are widely used in modelling of nonsymmetric relations in various sciences and engineering disciplines. We introduce a new invariant of strongly connected directed graphs - minimal number of vertices or edges necessary to remove to make remaining graphs not strongly connected. By analogy with undirected graphs we call this invariant strong connectivity. We give first properties of this invariant. Computational results for some publicly available connectome graphs used in neuroscience are described.

Key words. directed graph, strongly connected graph, connectivity, connectome

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1. Introduction.

1.1. The subject of study. Directed graphs are widely used to model various objects and processes having nonsymmetric features. Directed paths and related notions of strong connectedness and strongly connected components may play important roles in applications. In this paper we introduce new invariants of strongly connected graphs which describe stability of strong connectedness with respect to vertex and edge removal operations. Strongly connected graphs having high values of these invariants may be considered to be stably strongly connected with respect to removal of vertices or edges. Therefore strong connectivities may be useful in applications.

As an example of possible application we compute these invariants for connectome graphs which are studied in neuroscience.

1.2. Background.

1.3. Review of related graph theory. In this subsection we list the necessary notions and notations from graph theory, see also [5].

1.3.1. Directed graphs. Let $\Gamma = (V, E)$ be a directed graph. An element $(u, v) \in E$ corresponds to the edge $u \rightarrow v$. We will use notions of reachability between vertices, strong connectedness relation in the set of vertices, strongly connected

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components (SCC). By factoring out SCC one gets an acyclic quotient graph called *condensation graph*.

The underlying undirected graph of a directed graph Γ is denoted by $\mathcal{U}(\Gamma)$: nonempty sets of directed edges between any vertices u and v are substituted by undirected edges $u - v$.

1.3.2. Undirected graphs. Let $\Gamma = (V, E)$ be a noncomplete undirected graph. We will denote vertex/edge connectivity of Γ by $\zeta_0(\Gamma)/\zeta_1(\Gamma)$ (traditional notations: $\zeta_0 = \kappa$ and $\zeta_1 = \lambda$).

The directed graph obtained from an undirected graph Γ by substituting each undirected edge $u - v$ by two directed edges $u \rightarrow v$ and $u \leftarrow v$ is denoted as $\mathcal{D}(\Gamma)$. Note that $\mathcal{U}(\mathcal{D}(\Gamma)) = \Gamma$, but $\mathcal{D}(\mathcal{U}(\Gamma)) \neq \Gamma$, in general.

Given a vertex or edge subset A , we denote by $\Gamma - A$ the graph obtained by removing (erasing) A from Γ . Given a vertex subset U we denote by $\Gamma[U]$ the Γ -subgraph induced by U . These notations also make sense for directed graphs.

2. Main results.

2.1. Strong connectivity.

2.1.1. Definitions.

DEFINITION 2.1. Let $\Gamma = (V, E)$ with $|V| \geq 2$ be a strongly connected directed graph. We call a subset of vertices $U \subseteq V$ a *weakening vertex subset* provided $\Gamma - U$ is not strongly connected or $\Gamma - U$ has one vertex. We call a subset of edges $D \subseteq E$ a *weakening edge subset* provided $\Gamma - D$ is not strongly connected.

DEFINITION 2.2. Let $\Gamma = (V, E)$ with $|V| \geq 2$ be a strongly connected directed graph. Minimal cardinality of weakening vertex/edge subsets of Γ is called *strong vertex/edge connectivity (SVC/SEC)* of Γ (denoted as $\sigma_0(\Gamma)/\sigma_1(\Gamma)$).

2.1.2. Properties.

PROPOSITION 2.3. $\Gamma = (V, E)$ - a strongly connected directed graph, $|V| \geq 2$.

1. $\sigma_0(\Gamma) = \sigma_1(\Gamma)$.
2. $\sigma_0(\Gamma) \leq \zeta_0(\mathcal{U}(\Gamma))$.
3. If $\Gamma = \mathcal{D}(\Delta)$ for some undirected graph Δ , then $\sigma_0(\Gamma) = \zeta_0(\Delta)$.

Proof.

1. Consider Γ as a flow network - an edge-weighted graph with weight of each

edge equal to 1. For any two vertices u, v define $MF(u, v)$ equal to the maximal flow from u to v in Γ . $MF(u, v)$ is equal to the cardinality of a maximal system of disjoint (u, v) -paths. Define

$$\sigma(u, v) := \min(MF(u, v), MF(v, u)).$$

We prove that $\sigma_i(\Gamma) = \min_{u \in E, v \in E} \sigma(u, v)$.

Let u, v be vertices with $\sigma(u, v) = k$. By removing at most k vertices or edges we create a graph where there is no directed path between u and v in at least one direction (directed disjoint (u, v) or (v, u) -path systems are *disrupted*), thus u and v are in different SCC and $\sigma_i(\Gamma) \leq k$. Taking minimum over u, v we get $\sigma_i(\Gamma) \leq \min_{u, v} \sigma(u, v)$.

To prove the opposite inequality, start with Γ and remove a minimal weakening vertex or edge set of cardinality m . Then there exist 2 vertices u, v in different SCC such that $MF(u, v) \leq m$ or $MF(v, u) \leq m$ and thus

$$\min_{u, v} \sigma(u, v) \leq \min(MF(u, v), MF(v, u)) \leq m.$$

This proves the inequality $\min_{u, v} \sigma(u, v) \leq \sigma_i(\Gamma)$ and the statement.

2. Removal of a minimal $\mathcal{U}(\Gamma)$ -disconnecting vertex subset from Γ makes the remaining graph disconnected.
3. There are zero or two arrows in both directions between any two vertices of Γ . For any $U \subseteq V$ $\Gamma - U$ is strongly connected iff $\mathcal{U}(\Gamma) - U$ is connected hence the statement is proved.

□

DEFINITION 2.4. Let $\Gamma = (V, E)$ with $|V| \geq 2$ be a strongly connected directed graph. Minimal cardinality of weakening vertex or edge subsets of Γ is called strong connectivity (SC) of Γ (denoted as $\sigma(\Gamma)$). It follows from statement 1 of Proposition 2.3 that $\sigma(\Gamma)$ is well defined.

PROPOSITION 2.5.

1. Let $a, b \in \mathbb{N}$, $a \leq b$. There exists a directed graph $\Gamma_{a,b}$, such that $\sigma(\Gamma_{a,b}) = a$ and $\zeta_0(\mathcal{U}(\Gamma_{a,b})) = b$.
2. SC of a directed graph on n vertices can be computed in $O(n^5)$ time.

Proof.

1. For any $b \in \mathbb{N}$ $\zeta_0(K_{b+1}) = \sigma(\mathcal{D}(K_{b+1})) = b$. Thus the case $a = b$ is proved.

Let $a < b$. We consider two subcases: $a \leq \frac{b}{2}$ and $a > \frac{b}{2}$.

Case $a \leq \frac{b}{2}$.

$\Gamma_{a,b} = (V_{a,b}, E_{a,b})$, where $V_{a,b} = \{U, V, W_a, W'_{b-a}\}$, $|U| = |V| = b+1$, $|W_a| = a$, $|W'_{b-a}| = b-a$. $E_{a,b}$ contains 1) edges from every element of U to every

element of W_a , 2) edges from every element of W_a to every element of V , 3) edges from every element of V to every element of W'_{b-a} , 4) edges from every element of W'_{b-a} to every element of U .

In this case $a = \min(a, b - a)$, W_a is a minimal weakening vertex subset for $\Gamma_{a,b}$. $W_a \cup W'_{b-a}$ is a minimal disconnecting vertex subset of $\mathcal{U}(\Gamma_{a,b})$.

Case $a > \frac{b}{2}$.

$\Gamma_{a,b} = (V_{a,b}, E_{a,b})$, where $V_{a,b} = \{U, V, W_a, W'_{b-a}\}$, $|U| = |V| = b + 1$, $|W_a| = a$, $|W'_{b-a}| = b - a$. $E_{a,b}$ contains 1) edges in both directions between every element of U and every element of W_a , 2) edges in both directions between every element of W_a and every element of V , 3) edges from every element of U to every element of W'_{b-a} , 4) edges from every element of W'_{b-a} and every element of V .

In this case W_a is a minimal weakening vertex subset for $\Gamma_{a,b}$.

In both cases $W_a \cup W'_{b-a}$ is a minimal disconnecting vertex subset of $\mathcal{U}(\Gamma_{a,b})$ and $\zeta_0(\Gamma_{a,b}) = |W_a \cup W'_{b-a}| = b$.

2. Assume again that Γ is an edge-weighted graph with weight of each edge equal to 1. The minimal number of vertices necessary to remove to make u and v in different SCC (denoted by $\sigma_0(u, v)$) is equal to $\min(MF(u, v), MF(v, u))$, it can be computed in $O(n^3)$ time, see [4]. By statement 1 of this proposition we have that $\sigma(\Gamma) = \min_{u,v} \sigma_0(u, v)$. Since there are $\frac{n(n-1)}{2}$ vertex pairs, it follows that $\sigma(\Gamma)$ can be computed in $O(n^3) \cdot n^2 = O(n^5)$ time.

□

EXAMPLE 2.6. $\Gamma_{1,3}$ is shown in Fig.1. $W_1 = \{1\}$, $W'_2 = \{2, 3\}$. A minimal weakening vertex subset of $\Gamma_{1,3}$ is W_1 . A minimal disconnecting vertex subset of $\mathcal{U}(\Gamma_{1,3})$ is $W_1 \cup W'_2$.

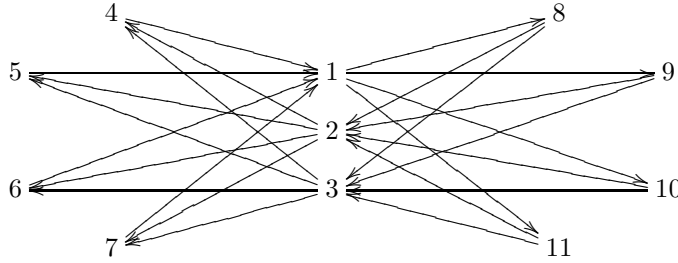


Fig.1. - $\Gamma_{1,3}$.

EXAMPLE 2.7. $\Gamma_{2,3}$ is shown in Fig.2. Pairs of oriented edges with common vertices are shown as undirected edges. $W_2 = \{1, 2\}$, $W'_1 = \{3\}$. A minimal weakening vertex subset of $\Gamma_{2,3}$ is W_2 . A minimal disconnecting vertex subset of $\mathcal{U}(\Gamma_{2,3})$ is $W_2 \cup W'_1$.

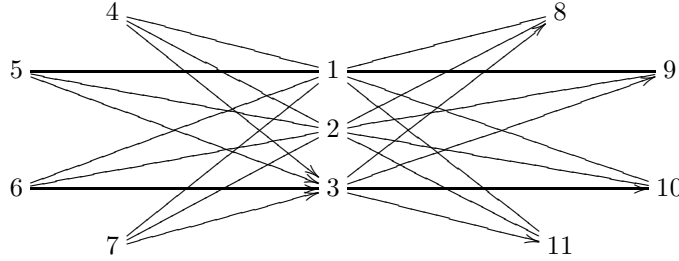


Fig.2. - $\Gamma_{2,3}$, pairs of opposite edges shown as undirected edges.

The next proposition shows that vertices and edges can be removed in any proportion and any order and the total number of removed elements is the only invariant.

PROPOSITION 2.8. $\Gamma = (V, E)$ - a strongly connected directed graph, $|V| \geq 2$.

1. For any $k \in \mathbb{Z}$, $0 \leq k \leq \sigma(\Gamma)$, there is a vertex subset $U \subseteq V$ and an edge subset $D \subseteq E$ such that
 - (a) $|U| = k$, $|D| = \sigma(\Gamma) - k$,
 - (b) $\Gamma - U - D$ is not strongly connected for any removing order.
2. For any $U' \subseteq V$ and $D' \subseteq E$ with $|U'| + |D'| < \sigma(\Gamma)$ we have that $\Gamma - U' - D'$ is strongly connected, for any removing order.

Proof.

There are two vertices u, v such that $\min(MF(u, v), MF(v, u)) = \sigma(\Gamma)$. There is a disjoint path system having $\sigma(\Gamma)$ elements with fixed source and sink vertices (u or v).

A disjoint path system of k paths with fixed source and sink vertices can be disrupted (all paths eliminated) by removing a vertex or an edge from each of k paths, in any order. Such a path system and any path system having more elements can not be disrupted by removing a subset of vertices and edges of cardinality less than k . Thus a vertex subset U and an edge subset D required in statement 1 exist and statement 2 is also shown true. \square

REMARK 2.9. Computation of SC can be iterated to get more information about graph structure as follows. Suppose we are given a strongly connected graph Γ .

1. Find all (or at least one) minimal weakening set, find all resulting SCC $\Gamma_{1,1}, \Gamma_{1,2}, \dots, \Gamma_{1,n_1}$ and the corresponding condensation graph Δ_1 with n_1 vertices.
2. Repeat Step 1 with each nontrivial SCC. Get a family of acyclic condensation graphs and a family of SCC.

2.2. An application. It makes sense to study strong connectivity in graph models where directed paths and reachability play important roles. A very simple example of such an application would be a city graph model where vertices are street intersections and edges are (possibly one-way) streets. It makes sense to ask how many one-way streets or intersections can be closed so that the remaining street network is still strongly connected.

In this subsection we describe some computational results related to strong connectivity of connectome graphs considered in neuroscience.

2.2.1. Connectome graphs. Connectome graphs are discrete mathematical models used for modelling nervous systems on different scales, see [6], [9]. On the microscale level these graphs are special cases of *cell graphs* which are studied for many types of human body tissues. In such graphs vertices correspond to cells and edges correspond to physical cell contacts or substance transfer (*volume transmission*, see [1]) links. See [2] for an example of cell graph application in tumour tissue modelling. On mesoscale and macroscale levels connectome graphs are essentially quotient graphs of microscale cell graphs. In this paper we do not deal with connectome scale, nature of graph edges and other modelling issues, we are interested only in applications of graph-theoretic concepts and algorithms. Connectome graph edges may be directed, undirected and weighted (labelled). We consider only directed graph structure of connectomes, each edge is assigned the constant weight 1. We use data and references available at [7].

We assume that direction of connectome edges, directed paths and reachability have considerable biological meaning. For example, direction of edges may be correlated with flow of signals or substances.

2.2.2. Cat. Graph (C) description - macroscale connectome graph (cortiocortical connections, see [8]), strongly connected graph with 65 vertices and 1139 edges, $\zeta_i(\mathcal{U}(C)) = 3$, diameter 3, minimal degree 3, maximal degree 45.

$\sigma(C) = 1$. There is a unique weakening vertex for C . After removing it C splits into one trivial SCC and a SCC having 63 vertices. There is a unique weakening edge pair for C . After removing it C splits into two SCC having 64 and 1 vertices.

We do 6 iterations. At each step the graph splits into one trivial and one large strongly connected component. The list of SC of large components starting from C is [1, 2, 3, 3, 3, 3, 2]. The list of vertex connectivities of underlying graphs of large components is [3, 3, 7, 7, 7, 6].

2.2.3. Rat. There are 3 macroscale connectomes $\{R_1, R_2, R_3\}$ for *Rattus norvegicus* available at [7]. Every graph has one nontrivial SCC having 502 or 493 vertices.

Graphs have similar properties in all cases. $\zeta_i(\mathcal{U}(R_n)) = 7$.

$\sigma(R_n) = 2$. Each graph has a unique weakening vertex pair. After removing a weakening vertex pair R_n splits into one or several trivial SCC and one large SCC.

2.2.4. Fly. A mesoscale fly connectome graph (see [10]) description - 1781 vertices and 9735 edges, 996 strongly connected components - one with 785 vertices, one with 2 vertices, the other components trivial, minimal degree 1, maximal degree 927.

Let F_1 - SCC of the described graph with 785 vertices. $\zeta_i(\mathcal{U}(F_1)) = 1$. $\sigma(F_1) = 1$, 173 weakening vertices. After removing a weakening vertex F_1 splits into several trivial and 1 large SCC. F_1 has 245 weakening edges.

2.3. Conclusion. We introduce study of a new invariant for directed graphs - strong connectivity. It can be computed in polynomial-time using network algorithms. Disruption of strong connectedness can be obtained by removing edges or vertices in any proportion and any order.

We present some computation results involving graph models in biology - connectome graphs of nerve tissues.

Some of our observations concerning connectome graphs:

- 1) after removing a minimal weakening vertex subset connectome graphs split into one nontrivial and one or several trivial SCC;
- 2) strong connectivities are close to undirected connectivities;
- 3) often there is a unique weakening vertex or edge subset.

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